

# Transition temperature of the homogeneous and dilute Bose gas in D-dimensions

Sang-Hoon Kim \*

Division of Liberal Arts and Sciences, Mokpo National Maritime University, Mokpo 530-729, R.O. Korea

The phase transition temperature of the homogeneous and dilute Bose gas in D-dimensions ( $2 \leq D \leq 3$ ) is calculated by a mean field-based statistical method. The shift of the phase transition temperature is written up to the leading order as  $\Delta T_c/T_c^0 = c\gamma^\alpha$ , where  $\gamma = n^{1/3}a$ . We derived Huang's result of the phase transition temperature in the generalized dimensions. We show that  $c(D)$  is positive and  $\alpha(D) = 2(D/2 - 1)^2$  in the short-wavelength range. The origin of the difference between  $\alpha = 1/2$  and  $\alpha = 1$  at  $D=3$  is discussed. The  $T_c$  at  $D = 2$  is calculated in the same scheme. The result is compared with Fisher and Hohenberg's KT temperature.

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## I. INTRODUCTION

The homogeneous and dilute(HD) Bose gas is a fundamental topic of physics to which a lot of theoretical efforts have been devoted in recent years. The shift of the transition temperature( $T_c$ ) remains an interesting problem to this day. At the same time, the study of  $T_c$  in two-dimensions(2D) has been a crucial topic because it is directly connected with the Kosterlitz-Thouless(KT) transition.

In a homogeneous system, the density in D-dimensions is expressed as  $n(D) = n(1)^D$ . The diluteness is expressed by a dimensionless gas parameter  $\gamma = n(1)a = n(3)^{1/3}a$ , where  $n(1)$  is the one-dimensional number density and  $a$  is the  $s$ -wave scattering length. In a repulsive and dilute gas,  $\gamma$  is positive and much less than 1. The *order of diluteness* can be defined as  $|\ln \gamma|$ , and much larger than 1. The argument in the paper is valid only for an extremely weakly interacting Bose system, and therefore cannot be applied to a strongly interacting system such as real liquid  $^4\text{He}$ .

The transition temperature of the non-interacting homogeneous Bose gas in D-dimensions is [1]

$$T_c^0(D) = \frac{2\pi\hbar^2}{mk_B} \frac{n(2)}{\zeta(\frac{D}{2})^{\frac{2}{D}}}, \quad (1)$$

where  $2 \leq D \leq 3$  and  $\zeta$  is the Riemann-zeta function. The transition temperature of the HD Bose system is written as a series form of  $\gamma$ . The leading orders in 3D and 2D are

$$T_c(3) = T_c^0(3)(1 + c\gamma^\alpha + \cdots), \quad (2)$$

$$T_c(2) = T_c^0(3)(0 + \frac{c'}{|\ln \gamma|^{\alpha'}} + \cdots). \quad (3)$$

The difficulty in finding the  $c$  comes from a breakdown of perturbation theory at the second order transition temperature. Comparing with Eq. (2), the interacting system is equivalent to using the effective mass

$m^* = m/(1 + c\gamma^\alpha)$  in the ideal gas expression for  $T_c^0(3)$ . The positive  $c$  means that the effective mass of the quasi-particle is reduced and vice versa for the negative  $c$ .

Two choices of  $\alpha$  for the leading order have been determined with a repulsive hard-sphere interaction  $\alpha = 1/2$ , suggested by Toyota and Huang from mean field-based microscopic theories [2, 3, 4]. On the other hand,  $\alpha = 1$  was suggested by others [5, 6, 7, 8, 9]. Toyota and Huang reported the coefficient  $c$  as

$$|c| = \frac{8\sqrt{2}\pi}{3\zeta(\frac{3}{2})^{\frac{2}{3}}} = 3.524, \quad (4)$$

but the others claimed that  $c$  is on the order of 1. Nowadays,  $\alpha = 1$  and positive  $c$  has been widely accepted since numerical simulations [8, 9].

Fisher and Hohenberg(FH)'s paper has suggested a KT transition temperature in 2D for the HD Bose gas from the Bogoliubov theory under the condition that  $\ln \ln \gamma^{-2} \gg 1$ . It has a double logarithmic form as [10]

$$T_c^{FH}(2) = \frac{2\pi\hbar^2}{mk_B} \frac{n(2)}{\ln \ln \gamma^{-2}}. \quad (5)$$

At a glance, FH's expression looks different from the Eq. (3).

In this paper, we will calculate the  $\alpha(D)$  and  $c(D)$  of the HD Bose gas between 2 and 3 dimensions with a semi-classical method. We will explain why Toyota and Huang's leading order  $\alpha = 1/2$  is different from others where  $\alpha = 1$ . The KT transition temperature in 2D will be obtained under the same scheme and compared with Hohenberg and Fisher's, too.

## II. EQUATION OF STATES IN D-DIMENSIONS

For a momentum-independent interaction, the mean field contribution to the self-energy  $\Sigma(D)$  is  $n(D)U(D)$ [11], where  $U(D)$  is the D-dimensional coupling constant. Then, the dispersion relation of the weakly interacting Bose gas in D-dimensions from the

\*shkim@mmu.ac.kr

mean field theory is

$$\varepsilon = \varepsilon_0 + \Sigma(D), \quad (6)$$

where  $\varepsilon_0 = p^2/2m$ . It corresponds to a short-wavelength range in the Bogoliubov energy spectrum [11], and it help us to calculate  $T_c$  without any breakdown problem. In principle, this scheme draws from Hartree-Fock theory, which assumes that the atoms behave as non-interacting bosons in a self-consistent mean field [12, 13]. We may choose the self-energy as the exact form  $2nU$  instead of  $nU$  because the two mean field contributions to the self-energy from Hartree and Fock are equal. However, for convenience we will use Eq. (6) because the factor of 2 is out of focus here.

The  $U(D)$  is obtained from a D-dimensional scattering theory. Under the hard-sphere interaction of the diameter  $a$ , it is well-known in 3D as

$$U(3) = \frac{4\pi\hbar^2 a}{m}. \quad (7)$$

In 2D, on the other hand, it has a logarithmic form [14, 15]

$$U(2) = \frac{2\pi\hbar^2}{m} \frac{1}{\ln(1/ka)} = \frac{4\pi\hbar^2}{m} \frac{1}{|\ln \gamma^2|}. \quad (8)$$

However, we had better use the D-dimensional interpolated form of  $U(D)$  between two and three dimensions for general purposes [16].

$$U(D) = \frac{4\pi^{\frac{D}{2}} \hbar^2 a^{D-2}}{\Gamma(\frac{D}{2} - 1) m}. \quad (9)$$

The average occupation number of the momentum-independent potential is written from Eq. (6) as

$$\begin{aligned} n_p &= \frac{1}{z^{-1} e^{\beta(\varepsilon_0(p) + \Sigma)} - 1} \\ &= \frac{1}{z_e^{-1} e^{\beta\varepsilon_0(p)} - 1}, \end{aligned} \quad (10)$$

where  $\beta = 1/k_B T$ . The  $z$  is the fugacity given by  $z = e^{\beta\mu}$  and 1 below  $T_c$ . The  $\mu$  is the chemical potential, which is 0 below the  $T_c$  and negative above the  $T_c$ . The  $z_e$  is the effective fugacity given by

$$z_e = e^{-\beta(|\mu| + \Sigma)} \equiv e^{\beta\mu_e}. \quad (11)$$

Note that  $0 < z_e < 1$ . The effective chemical potential  $\mu_e$  is written in the same way:  $\mu_e = \mu - \Sigma$ . Therefore, the interacting system has the same formula as the ideal system except for the effective chemical potential and the effective fugacity.

Let us define the ratio between the effective fugacity and the fugacity as

$$\eta = \frac{z_e}{z} = e^{-\beta\Sigma}. \quad (12)$$

Note that  $0 < \eta_c < 1$  and 1 for the ideal system. As  $T \rightarrow T_c$ , then  $z_e \rightarrow e^{-\beta_c \Sigma} \equiv \eta_c$ . When the interaction is very weak, the phase transition is described by the variable:  $\eta_c \simeq 1 - \beta_c \Sigma$ . Therefore, it is clear that even for a momentum-independent potential, it shifts the  $T_c$ .

From Eq. (10) we have the D-dimensional equation of states with  $z_e$  as [17]

$$n(D)\lambda^D = g_{\frac{D}{2}}(z_e). \quad (13)$$

$\lambda = \sqrt{2\pi\hbar^2/mk_B T}$  is the thermal wavelength, and  $g_s(z) = \sum_{l=1}^{\infty} z^l/l^s$  is the Bose function. In this way we can write the transition temperature in D-dimensions,  $2 \leq D \leq 3$ , of the interacting system as

$$T_c(D) = \frac{2\pi\hbar^2}{mk_B} \frac{n(2)}{g_{\frac{D}{2}}(\eta_c)^{\frac{2}{D}}}. \quad (14)$$

From Eq. (13) the relation of the transition temperature between ideal and interacting systems is written as

$$(T_c^0)^{\frac{D}{2}} g_{\frac{D}{2}}(1) = T_c^{\frac{D}{2}} g_{\frac{D}{2}}(\eta_c). \quad (15)$$

As  $D$  approaches 2, the above quantity approaches a finite value,  $2\pi\hbar^2 n(2)/mk_B$ , from Eq. (1). It is about 5.92K for liquid  $^4\text{He}$ .

### III. $T_c$ BETWEEN 2 AND 3 DIMENSIONS

From Eq. (15), it is necessary to expand the  $g_{\frac{D}{2}}(\eta_c)$  in an analytic form to find the transition temperature. Since  $\beta_c \Sigma$ , the interaction term at  $T_c$ , is much less than 1, we use the expansion of the Bose functions for  $D > 2$  [18].

$$g_{\frac{D}{2}}(e^{-\beta_c \Sigma}) \simeq \zeta\left(\frac{D}{2}\right) + \Gamma\left(1 - \frac{D}{2}\right) (\beta_c \Sigma)^{\frac{D}{2}-1} + \mathcal{O}(\beta_c \Sigma). \quad (16)$$

Substituting Eq. (16) into Eq. (15),

$$\begin{aligned} T_c^{\frac{D}{2}} &= (T_c^0)^{\frac{D}{2}} \frac{\zeta(\frac{D}{2})}{g_{\frac{D}{2}}(\eta_c)} \\ &= (T_c^0)^{\frac{D}{2}} \left[ 1 + \frac{\Gamma(1 - \frac{D}{2})}{\zeta(\frac{D}{2})} (\beta_c \Sigma)^{\frac{D}{2}-1} \right]^{-1}. \end{aligned} \quad (17)$$

Up to the linear term, we have

$$T_c \simeq T_c^0 \left[ 1 - \frac{2}{D} \frac{\Gamma(1 - \frac{D}{2})}{\zeta(\frac{D}{2})} (\beta_c^0 \Sigma)^{\frac{D}{2}-1} \right]. \quad (18)$$

From Eqs. (1) and (6)

$$\begin{aligned} \beta_c^0(D)\Sigma(D) &= \frac{m\zeta(\frac{D}{2})^{\frac{2}{D}} n(1)^{D-2} U(D)}{2\pi\hbar^2} \\ &= \frac{2\pi^{\frac{D}{2}-1} \zeta(\frac{D}{2})^{\frac{2}{D}}}{\Gamma(\frac{D}{2} - 1)} \gamma^{D-2}. \end{aligned} \quad (19)$$

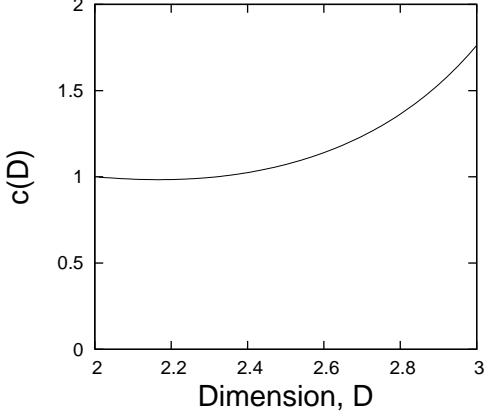


FIG. 1:  $c(D)$  as a function of dimensions.

Note that  $\beta_c^0(3)\Sigma(3) = 2\zeta(\frac{3}{2})^{\frac{2}{3}}\gamma$ . The leading order  $\alpha$  of the shift of the chemical potential  $\Sigma(3)/k_B$  is 1, but it definitely cannot be the shift of  $T_c(3)$ .

Substituting Eq. (19) into Eq. (18), we obtain the D-dimensional  $T_c$  as

$$T_c(D) = T_c^0(3) \left[ 1 - \frac{2^{\frac{D}{2}} \pi^{\frac{(D-1)^2}{2}} \Gamma(1 - \frac{D}{2})}{D \zeta(\frac{D}{2})^{\frac{2}{D}} \Gamma(\frac{D}{2} - 1)^{\frac{D}{2}-1}} \gamma^{2(\frac{D}{2}-1)^2} \right]. \quad (20)$$

Therefore, we have the two coefficients of the  $T_c$  shift in D-dimensions as

$$c(D) = -\frac{2^{\frac{D}{2}} \pi^{\frac{(D-1)^2}{2}} \Gamma(1 - \frac{D}{2})}{D \zeta(\frac{D}{2})^{\frac{2}{D}} \Gamma(\frac{D}{2} - 1)^{\frac{D}{2}-1}} \quad (21)$$

and

$$\alpha(D) = 2 \left( \frac{D}{2} - 1 \right)^2, \quad (22)$$

where  $2 < D \leq 3$ . Note that  $\alpha = 0$  in 2D as the Eq. (3).

There has been a long dispute about the sign of the  $c$ . From Eq. (21) we see that it is decided by  $-\Gamma(1 - \frac{D}{2})$ . Therefore, the  $c$  is positive at  $D > 2$ . It is plotted in FIG. 1. However, note that the minimum of  $c(D)$  is not at  $D = 2$  but at  $D = 2.17$ . If  $D = 3$ , we obtain  $c(3) = 4\sqrt{2}\pi/3\zeta(3/2)^{2/3} = 1.762$  and  $\alpha = 1/2$ . Note that  $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ . The  $c(3)$  is exactly half of Toyota and Huang's microscopic results in Eq. (4) in that they chose  $2nU$  instead of  $nU$  as the self-energy. The positive  $c$  means that the BEC temperature,  $T_c$ , is different from the lambda transition temperature,  $T_\lambda$ , and there possibly exists a pseudogap-like region between  $T_c$  and  $T_\lambda$  in superfluidity.

#### IV. $T_c$ IN TWO-DIMENSIONS

For an ideal Bose system, there is no condensation because a long-wavelength phase fluctuation destroys the

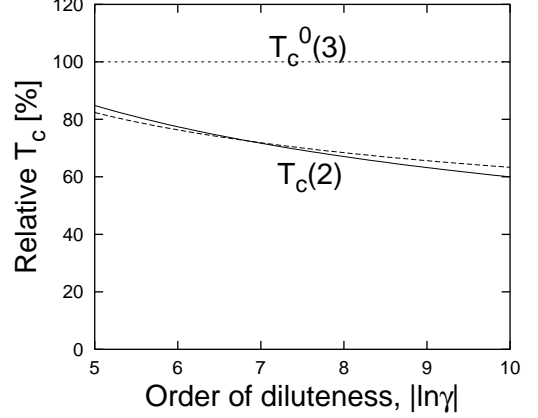


FIG. 2: The relative transition temperature in 2D compared with that of the 3D ideal gas. The solid line is our  $T_c(2)$  and the dashed line is  $T_c^{FH}(2)$  by Fisher and Hohenberg.

long-range order. On the other hand, in an interacting system, the interparticle interactions drive a phase transition to a condensate state and then, there exists a nonzero condensate density above a specific temperature  $T_c^{KT}(2)$ .

We can also obtain the transition temperature in 2D from Eq. (14) as

$$T_c(2) = \frac{2\pi\hbar^2}{mk_B} \frac{n(2)}{g_1(\eta_c)}. \quad (23)$$

The convergence of  $g_1(\eta_c)$  is extremely slow. Therefore, we need to use another method for the limiting value. The Euler's constant  $C = 0.5772\dots$  is defined as

$$\lim_{s \rightarrow 1} \left[ \zeta(s) - \frac{1}{s-1} \right] = C. \quad (24)$$

It is effective up to the first order even when  $\lim_{x \rightarrow 1} g_1(x)$ . Then, the limiting behavior of the  $g_1(\eta_c)$  when  $\eta_c$  is close to 1 is

$$g_1(\eta_c) \simeq \frac{1}{1 - \eta_c} = \frac{1}{\beta_c(2)\Sigma(2)}. \quad (25)$$

Substituting Eq. (25) into Eq. (23),

$$T_c(2) = \frac{2\pi\hbar^2 n(2)^2 U(2)}{mk_B^2 T_c(2)}. \quad (26)$$

Substituting Eq. (8) into Eq. (26), we obtain the 2D transition temperature as

$$T_c(2) = \frac{2\pi\hbar^2}{mk_B} \frac{n(2)}{\sqrt{|\ln \gamma|}}. \quad (27)$$

The leading orders in Eq. (3) are obtained as  $\alpha' = \frac{1}{2}$ , and  $c' = \zeta(\frac{3}{2})^{\frac{2}{3}} = 1.897$ .  $T_c(2)$  is shown in FIG. 2, and compared with  $T_c^{FH}(2)$  and  $T_c^0(3)$  in the range of interest.

FH follows the approximation for  $T_c^{FH}(2)$  under the *ansatz* :  $\mu \sim n/|\ln(a^2\mu)|$  [10]. Then, the transition temperature in 2D is proportional to the following :

$$T_c^{FH}(2) \sim \frac{n(2)}{|\ln ka|} \sim \frac{n(2)}{\ln(\mu a^2)} \sim \frac{n(2)}{\ln \ln \gamma^{-2}}. \quad (28)$$

We think the *ansatz* is not necessary, because FH's transition temperature is nothing but  $T_c^{FH}(2) = \Sigma(2)/2k_B$  or just the chemical potential in 2D. Note that the factor of half comes from choosing the self-energy  $nU$  instead of  $2nU$ . This happens because the clean form of  $U(2)$  in Eq. (8) was not known at that time.

## V. SUMMARY

We derived the transition temperature of the HD Bose gas between 2 and 3 dimensions in the short-wavelength range using a semi-classical method. The two coefficients of the leading order of  $T_c(D)$  ( $\alpha$  and  $c$ ), where  $2 < D \leq 3$ , are obtained as a function of the dimensions. Toyota and

Huang's results are just the special case of our calculation at  $D = 3$ . It was obtained not because the mean field theory does not work, but because the long-wavelength limit is not included. Therefore, for an extremely low density and at the short-wavelength range, we may accept Toyota and Huang's results of  $\alpha = 1/2$  [3, 4].

The KT transition temperature in 2D in the same condition is obtained. The two coefficients of the leading order of  $T_c(2)$  ( $\alpha'$  and  $c'$ ) are obtained, too. We show that Fisher and Hohenberg's KT transition temperature is just the chemical potential in 2D. The leading orders of the  $T_c$  in 3D and 2D are obtained as  $\alpha = \alpha' = 1/2$  in the short-wavelength range, but the reliable results of  $\alpha = \alpha' = 1$  are expected if we include the long-wavelength limit.

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